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New Measure Representing Degree of Controllability for Disturbance Rejection

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I. Introduction

A DEGREE of controllability (DOC) represents how controllable a given system is, qualitatively or quantitatively. DOC has been dealt with in the literature since Kalman et al. [1] first discussed it. Hughes and Skelton [2] proposed a controllability norm that gives detailed information on the rank of the controllability matrix of a system. Hamdan and Nayfeh [3] extended the Popov–Belevitch–Hautus test to define measures of modal controllability. Tarokh [4] introduced controllability measures related to frequency-domain characteristics such as zeros and residues. Viswanathan et al. [5] proposed the minimum of 2-norms of initial conditions that can be returned to zero with a control input for which the infinite norm is not greater than 1 for the degree of controllability. Though it can be viewed as a parameter that quantitatively describes the controllability of the system, its closed-form expression is not available and requires an approximation to obtain its value. Müller and Weber [6] presented three candidates of DOC based on the scalar measures of the controllability Grammian matrix for a linear continuous system. The measures are related to the minimum input energy that is required to regulate a system from initial conditions in a finite time interval. If a system requires smaller input energy for its regulation from initial conditions than others, it can be considered more controllable. Roh and Park [7] proposed a novel concept titled modal degree of controllability (MDOC) that represents the relative performance of a specific candidate set with a predetermined number of actuators, compared with the performance achievable with the full set of actuators. The MDOC is defined as a ratio between the two minimum input energies required to regulate a system from initial modal disturbances to zero with a specific candidate set and the full set of actuators. However, these works for DOC do not take the effect of external disturbance into account directly.

Naturally, research on the measures for disturbance rejection has followed. The degree of controllability for disturbance rejection is of interest because disturbance rejection is often the main objective of process control [8]. Some plants have better built-in disturbance-rejection capabilities than others; that is, their performance with respect to disturbance rejection is better. Unquestionably, it has been known for a long time that disturbance rejection is an important property of a plant. Stanley et al. [9] introduced a dimensionless measure for disturbance that they called relative disturbance gain, and it can be calculated from steady-state information only for a square plant G. Morari [10] took account of the magnitude of the inputs required for disturbance rejection and argued that the minimum singular value of the plant may provide a useful measure. In addition, he qualified the effect of model/plant mismatch on control quality. However, he did not use any information about the disturbance model. Morari at al. [11] considered the allowed magnitude of disturbances to achieve feasible operation in the steady state and denoted their measure as the resilience index. However, the resilience index does not take into account the different possible directions of the disturbances, but only their size. They also used singular values to bound the magnitude of disturbances under constraints on the manipulated variables. Shimizu and Matsubara [12] discussed the direction of combined disturbances in the frequency domain using the singular value decomposition. Skogestad and Morari [13] proposed a similar analysis, but also considered the direction of an individual disturbance. They stressed that in multivariable systems, some disturbances may be difficult to reject if they are in the undesirable direction compared with the direction of the plant and to quantify this, they introduced the disturbance condition number. However, they analyzed the direction of disturbances for a square transfer matrix G, and the measure depends only on the direction of the disturbance, but not on its magnitude. Skogestad and Hovd [14] and Hovd and Skogestad [15] argued that for decentralized control, one should use the closed-loop disturbance gain when evaluating the effect of disturbances. The measures described previously have a significant drawback in that the number of input variables is at least equal to the number of outputs to be controlled.

Luyben [16] stressed that the choice of the control structure may strongly influence the sensitivity to disturbances and pointed out that in many processes there is an eigenstructure (i.e., an intrinsically self-regulating control structure). Cao and Rossiter [17] also proposed the input-disturbance alignment (IDA). The IDA is defined as the projection norm of the transfer function from a scalar disturbance to output g_d on the range of the transfer function from input to output G. It represents how aligned g_d is with G. However, the IDA only takes into account the disturbance direction without considering the magnitude of that disturbance. To overcome such a limitation of IDA, Cao et al. [18] proposed the worst-case input-disturbance gain (WCIDG) and the input-disturbance gain deviation (IDGD). When the least-squares input required to keep the output norm as close as possible to zero is yielded as $u = G^{+}G_d d$, the WCIDG is defined as the norm of the ith row of $G^{+}G_d$ in the worst case. In addition, the IDGD is the maximum gain from a disturbance to the ith least-squares input deviation that is defined as the difference between the least-squares input $u = G^{+}G_d d$ and least-squares solution after eliminating the ith input channel. Thus, input having a relatively large WCIDG or IDGD value implies that this input plays an important role in disturbance rejection. Actually, the WCIDG and IDGD can be useful measures for input screening tools. Though they can be employed to determine the locations of actuators, they are not appropriate to represent the capabilities of the disturbance rejection of a given system. Most of the existing measures defined in the frequency domain have the advantage of representing the frequency-dependent capabilities for disturbance rejection in a theoretical point of view. Although they can evaluate the dynamic effects of disturbances in the system, the measure values at steady state are used in the literature. This results from the drawback that even if the measures can show the capabilities for disturbance rejection at each frequency component, they cannot have a representative value corresponding to the frequency spectrum. Mirza and Niekerk [19] used the size of the disturbance-sensitivity Grammian for the closed-loop system to indicate the effect of a disturbance. Based on the fact that a large disturbance-sensitivity Grammian will indicate that the disturbance has a large effect on the system, they determined the optimal actuator locations.

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In this Note, a new measure to represent the capabilities of disturbance rejection will be proposed. The new measure does not impose any restrictions on the amount of input and output. In addition, because the proposed measure is calculated from open-loop systems to qualify the intrinsic disturbance-rejection capability of the given systems, it does not depend on controller gains or control strategy. If the controller is determined, it can be augmented into the system, of course. The proposed measure can also consider the effect of disturbance magnitude on the disturbance rejection. The measure will be derived simply by using a controllability Grammian and a disturbance-sensitivity Grammian for the stochastic disturbance and will represent the capabilities for disturbance rejection quantitatively with a physically meaningful value: control energy.

II. Controllability and Disturbance Sensitivity Grammian

Controllability and disturbance-sensitivity Grammians will be discussed in this section. Let us consider a linear time-invariant system:

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw(t) \tag{1}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^r$, and $w(t) \in \mathbf{R}^l$ are the state, control input and disturbance vectors, respectively, and A, B, and D are constant matrices with appropriate dimensions. The disturbance is assumed to be Gaussian white noise with the known correlation function

$$R_w(\tau) = E[w(t)w^T(t+\tau)] = S_w\delta(\tau) \tag{2}$$

and mean

$$\mu = E[w(t)] = 0 \tag{3}$$

Let $\Phi(t, t_0)$ denote the state transition matrix associated with A. The controllability Grammian is defined as

$$W(t) = \int_0^t \Phi(t,\tau)BB'\Phi'(t,\tau)\mathrm{d}\tau \quad \text{for some } 0 < t < \infty \tag{4}$$

The controllability Grammian can be calculated by solving the following differential equation:

$$\dot{W}(t) = AW(t) + W(t)A' + BB' \tag{5}$$

Similarly, the disturbance-sensitivity Grammian can be defined,

$$\sum(t) = \int_t^t \Phi(t,\tau)DS_wD'\Phi'(t,\tau)\mathrm{d}\tau \quad \text{for some } 0 < t < \infty \tag{6}$$

satisfying the following differential equation:

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + DS_wD' \tag{7}$$

III. Measure for Degree of Controllability for Disturbance Rejection

A. Expected Minimum-Energy Transfer

The measure to be proposed here follows the ideas of Kalman et al. [1]. Even though they did not consider the effect of external disturbances, we are interested in rejecting disturbances with stochastic properties. Thus, the expected minimum transfer energy is considered to be the measure for the disturbance-rejection capability of the given system in the form of Eq. (1):

$$\text{Minimize } \rho = E\left[\int_0^T u'(t)u(t)\mathrm{d}t\right] \tag{8}$$

subject to $x(0) = 0$ and $x(T) = 0$. The initial condition is assumed to be zero because we focus on external disturbance alone without considering the stabilizing performance from the given initial conditions. The solution can be given by

$$u(t) = -B'e^{A'(T-t)}W(T)^{-1}\int_0^T e^{A(T-s)}Dw(s)\mathrm{d}s \tag{9}$$

for some $0 \le t \le T$. Readers interested in deriving the solution should see the details in [20]. The input covariance matrix can be found by substituting the solution of control input (9) into the definition of the covariance matrix:

$$\mathbf{R}_u(t) = E[u(t)u'(t)] = E\Bigg[B'e^{A'(T-t)}W_c^{-1}(T)\int_0^T e^{A(T-s)}Dw(s)\mathrm{d}s \cdot \int_0^T w(p)D'e^{A'(T-p)}\mathrm{d}pW_c^{-1}(T)e^{A(T-t)}B\Bigg] \tag{10}$$

Rearranging terms and moving the expected value with the integrals yields

$$\mathbf{R}_u(t) = B'e^{A'(T-t)}W^{-1}(T)\int_0^T\int_0^T e^{A(T-s)}DE[w(s)w(p)] \times D'e^{A'(T-p)}\mathrm{d}p\mathrm{d}s \cdot W_c^{-1}(T)e^{A(T-t)}B \tag{11}$$

Then using Eq. (2) and the definition of the Dirac delta function, Eq. (11) reduces to

$$\mathbf{R}_u(t) = B'e^{A'(T-t)}W_c^{-1}(T)\int_0^T e^{A(T-s)}DS_wD'e^{A'(T-s)}\mathrm{d}s \cdot W_c^{-1}(T)e^{A(T-t)}B \tag{12}$$

Using the solution of Eq. (6), the input covariance matrix can be expressed simply as

$$\mathbf{R}_u(t) = B'e^{A'(T-t)}W^{-1}(T)\Sigma(T)W^{-1}(T)e^{A(T-t)}B \tag{13}$$

As mentioned previously, the expected total transfer energy is the proposed measure representing the degree of controllability for disturbance rejection. The expected total energy can be calculated by integrating Eq. (13):

$$\int_0^T E[u'(t)u(t)]\mathrm{d}t = \mathrm{tr}\left\{\int_0^T E[u(t)u'(t)]\mathrm{d}t\right\} = \mathrm{tr}\left\{\int_0^T B'e^{A'(T-t)}W^{-1}(T)\Sigma(T)W^{-1}(T)e^{A(T-t)}B\mathrm{d}t\right\} \tag{14}$$

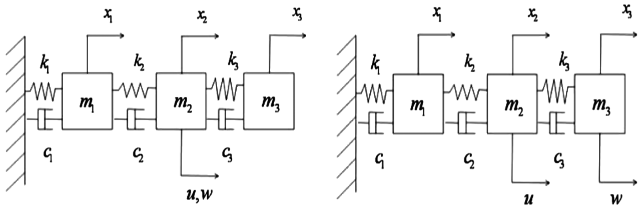
where $\mathrm{tr}\{\cdot\}$ means the summation of the diagonal terms of $\{\cdot\}$. Applying the property of the trace and rearranging terms, Eq. (14) yields

$$\begin{aligned}\rho &= \mathrm{tr}\left\{\int_0^T W^{-1}(T)\Sigma(T)W^{-1}(T)e^{A(T-t)}BB'e^{A'(T-t)}\mathrm{d}t\right\} \\ &= \mathrm{tr}\left\{W^{-1}(T)\Sigma(T)W^{-1}(T)\int_0^T e^{A(T-t)}BB'e^{A'(T-t)}\mathrm{d}t\right\} \\ &= \mathrm{tr}\{W^{-1}(T)\Sigma(T)W^{-1}(T)W(T)\} = \mathrm{tr}\{W^{-1}(T)\cdot\Sigma(T)\}\end{aligned} \tag{15}$$

As shown in Eq. (15), the measure can be calculated by solving the two differential equations, and their solutions are in the form of a Grammian matrix. As mentioned, the measure has a physically meaningful value of input energy with absolute units. Thus, it cannot only quantitatively represent the degree of controllability for the disturbance rejection of a given system, but it can also compare the capabilities for the disturbance rejection of systems with different setups.

B. Measure at Steady State

As shown in Eq. (15), the solutions of two differential equations depend on the final time T. As a result, the proposed measure will depend on the time T, which, to a large extent, has to be selected arbitrarily. To eliminate this dependency of the measure on T, we consider steady-state solutions of Eqs. (5) and (6), satisfying Eqs. (16) and (17) for asymptotically stable systems:



a) Matched disturbance b) Unmatched disturbance

Fig. 1 Three-degree-of-freedom mass-spring-damper system.

$$A\bar{W} + \bar{W}A' + BB' = 0 \tag{16}$$

$$A\bar{\Sigma} + \bar{\Sigma}A' + DS_w D' = 0 \tag{17}$$

Equations (16) and (17) are Lyapunov equations. Therefore, the measure for disturbance rejection at steady state can be expressed as

$$\rho = \int_0^\infty \mathbf{E}[u'(t)u(t)]\mathrm{d}t = \mathrm{tr}\{\bar{W}^{-1}\cdot\bar{\Sigma}\} \tag{18}$$

The measure can be solved easily by solving the two Lyapunov equations. Because the closed-form solutions for the Grammians exist, the method is not computationally intensive.

IV. Numerical Example

To demonstrate the usefulness of the proposed measure, a numerical example will be given. Let us consider a mass-spring-damper system, as shown in Fig. 1. To evaluate the proposed measure, we will consider the cases when the disturbance is matched and unmatched. Next, we will identify the variation of the measure according to the variation of a model parameter.

A. Mass-Spring-Damper System

It should be noted that because the number of output variables is more than that of input variables, the existing measures suggested in the frequency domain cannot be employed. To evaluate the performance of the proposed measure, we will consider two cases. One is when the disturbance is matched and the other is when it is unmatched. First, we will deal with the case of the matched disturbance. The state-space representation of the given model can be expressed as

$$\dot{x} = Ax + Bu + Dw \tag{19}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Then $x(t) \in \mathbf{R}^6$ and $u(t) \in \mathbf{R}^1$ are the state and control input, respectively; $w(t) \in \mathbf{R}^1$ is a disturbance with covariance $E[w(t)w^T(t)] = 1$; and A, B, and D are constant matrices with appropriate dimensions. The system is controllable and asymptotically stable. We first study the case when the matching condition is satisfied; that is, the disturbance is in the range of input matrix B. For the matched disturbance, the matrix D is given by

$$D = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]' \tag{20}$$

The system is depicted in Fig. 1a. Using Eqs. (16–18), the value of the proposed measure is yielded:

$$\rho = tr\{\bar{\Sigma}\cdot\bar{W}^{-1}\} = 6 \tag{21}$$

Next, we will calculate the measure for the case of the unmatched disturbance. Intuitively, it can be expected that disturbance rejection for an unmatched disturbance is more difficult than that for the matched disturbance. Thus, the measure for an unmatched disturbance possibly has a larger value than that for the matched disturbance. For the unmatched disturbance, the matrix D is given by

$$D = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]' \tag{22}$$

The system with an unmatched disturbance is depicted in Fig. 1b. The proposed measure is

$$\rho = tr\{\bar{\Sigma}\cdot\bar{W}^{-1}\} = 234 \tag{23}$$

From Eq. (21) and (23) it is found that the measure for as unmatched disturbance has about a 40 times larger value than the one for a matched disturbance. Because the proposed measure means the input energy, it is known that rejecting the effect of an unmatched disturbance requires about 40 times more control energy than the matched case. The DOC for the disturbance rejection of the system would be significantly degraded when an unmatched disturbance exists.

B. Parameter Variation

To validate the performance of the proposed measure, we will perform further analysis on the same example. We will change the spring constant k_3 from zero to infinity while the unmatched disturbance is excited to the system, as shown in Fig. 1b. The results from changing the spring constant k_3 are shown in Table 1. From Table 1, it is revealed that if the spring constant is very small, the capability for rejecting disturbance decreases. As the spring constant becomes larger, on the other hand, the measure becomes smaller. That means that the capability for disturbance rejection improves. Those results are expected from the physical understanding. The small spring constant means that second mass m_2 and third mass m_3 are almost not linked to each other. Thus, the control input applied to mass m_2 cannot affect the movement of mass m_3. Accordingly, the disturbance cannot be rejected easily. When the spring constant k_3 is very large, the two masses m_2 and m_3 are connected tightly to each other. Therefore, the disturbance can be considered as a matched disturbance. Accordingly, the measure converges to that of the matched case of Eq. (21) as the spring constant increases to infinity.

Next, let us consider the effect of final time T on the proposed measure. The convergent trends of the proposed measure with respect to time are shown in Fig. 2. Because of the limited scale of the figure, only four cases in which $k_3 = 0.1, 1, 1e1$ and $1e2(\mathrm{N/m})$ are drawn in the figure.

It is known from the results that the magnitudes of input energies early on are much larger than at later times. This means that when one wants to reject the effect of a disturbance in a short time, the degree of controllability for disturbance rejection becomes significantly degraded. It is also shown that as the spring constant coefficient k_3 becomes larger, the proposed measure can be approximated by the steady-state value for smaller final time T. This results from the fact that as the disturbance-rejection capabilities of the system improve, the convergent rate of the proposed measure becomes faster. Thus, the convergent rate can also be a useful measure for disturbance-rejection capabilities.

Table 1 Proposed measure according to parameter variation

Spring constant k_3, Nm^{-1}	0	0.01	0.1	1	1e1	1e2	1e3	∞
Proposed measure	∞	2.7e7	4.7e3	234	12.6	6.5	6.05	6

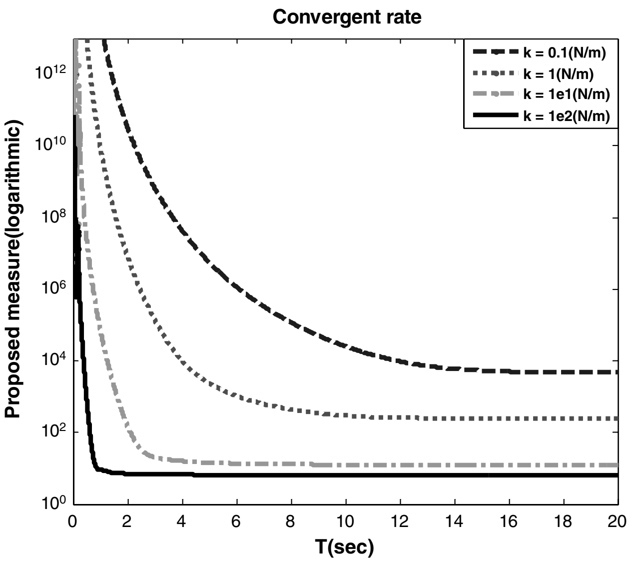


Fig. 2 Convergent rate of the proposed measure.

V. Conclusions

In this Note, we proposed a measure for the degree of controllability for disturbance rejection. The measure has a physical meaning of the average minimum energy required to make the state variable zero at a final time. For this measure, white noise with a zero mean is considered as an external disturbance. The approach depends on computation of the controllability Grammian and the disturbance-sensitivity Grammian. Those Grammian matrices can be obtained by solving two Lyapunov equations if the system is asymptotically stable and the final time is extended to infinity. Therefore, the computational simplicity is another advantage of the proposed measure. To demonstrate the usefulness of the proposed measure, we applied it to a simple 3-degree-of-freedom mass-spring-damper system. After considering matched and unmatched disturbances, respectively, and changing the parameter of spring constant k_3, it was shown that the proposed measure agrees well with physical interpretation. In addition, it was possible to compare the systems with totally different setups in terms of disturbance rejection, because the measure has an absolute unit of input energy. Further, this measure can be employed to the problem of determining optimal actuator locations for good disturbance rejection.

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